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NOTE

VIRTUAL MASS AND IMPULSE OF BUBBLE DISPERSIONS: REPLY TO A NOTE BY VAN WIJNGAARDEN

J. A. GEURST
Department of Mathematics and Informatics, Delft University of Technology, P.O. Box 356, 2600 AJ Delft, The Netherlands

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1. INTRODUCTION

The stability of two-phase flow models is a topic of current interest (Prosperetti & Satrape 1990). In Geurst (1985) the author developed a macroscopic theory of bubbly flow by starting from a properly extended form of Hamilton's variational principle. The inertial interaction of the bubbles associated with their virtual mass was taken into account explicity [see the review paper by Wallis (1989)]. Marginal stability could be achieved by taking a simple expression for the virtual-mass coefficient $m(\epsilon)$ as a function of the void fraction ϵ, viz.

$$m(\epsilon) = \tfrac{1}{2}\epsilon(1-\epsilon)(1-3\epsilon), \qquad [1]$$

in the case of spherical bubbles.

Kok (1988) showed that in a frame of reference moving with the volume velocity of the bubbly mixture—a reference frame being used by van Wijngaarden and his group at Twente University—the marginal stability criterion takes the still simpler form

$$k(\epsilon) = \tfrac{1}{2}\epsilon. \qquad [2]$$

Note that $k(\epsilon)$ is equal to the quantity $K^*(\epsilon)$ considered by van Wijngaarden (1991, this issue, pp. 809–814). For the proper definition of $k(\epsilon)$ see [19] below.

van Wijngaarden (1991), basing himself on Kok's result [2], argues that the stability criterion [1] should be related to the neglect of bubble interactions. It is demonstrated in section 3, however, that his argument relies on expressions for the virtual mass and impulse of a bubble dispersion that are not acceptable from a physical point of view. Before, in section 2, the system of macroscopic equations for bubbly flow is reviewed briefly. It is shown in section 4 that the analysis of non-linear concentration waves in Geurst & Vreenegoor (1988) provides a physical explanation of Kok's result.

2. MACROSCOPIC EQUATIONS FOR BUBBLY FLOW

The system of two-phase flow equations for a bubbly mixture comprises both kinematic and dynamic equations. The kinematic equations expressing, respectively, the conservation of mass of the two phases and the conservation of bubble number read in one-dimensional form

$$\frac{\partial \rho_i}{\partial t} + \frac{\partial}{\partial x}(\rho_i u_i) = 0 \quad (i = 1, 2) \qquad [3]$$

and

$$\frac{\partial n}{\partial t} + \frac{\partial}{\partial x}(n u_2) = 0. \qquad [4]$$

Here $\rho_1 = \rho_L(1-\epsilon)$ and $\rho_2 = \rho_G\epsilon$, while n represents the number density of the bubbles and u_i ($i = 1, 2$) denote the mass-averaged velocities of, respectively, the liquid and the gas phase. When dissipative effects are disregarded, the dynamic equations may be derived from a macroscopic form of Hamilton's variational principle (Geurst 1985, 1986; Geurst & Vreenegoor 1987). Let us assume that: (i) the liquid is incompressible; (ii) the flow proceeds isothermally; and (iii) the bubbles are nearly spherical. In that case the dynamic equations take the following form (cf. Wallis 1989, 1990):

$$\frac{\partial \pi_1}{\partial t} + \frac{\partial}{\partial x}[u_1\pi_1 + \tfrac{1}{2}m'(\epsilon)(u_2 - u_1)^2 - \tfrac{1}{2}u_1^2] = -\frac{1}{\rho_L}\frac{\partial p^*}{\partial x} + F_1, \qquad [5]$$

for the macroscopic motion of the continuous liquid phase; and

$$\frac{\partial \pi_2}{\partial t} + \frac{\partial}{\partial x}(u_2\pi_2 - \tfrac{1}{2}u_2^2) = -\frac{1}{\rho_G}\frac{\partial p^*}{\partial x} + F_2, \qquad [6]$$

for the motion of the disperse gas phase. Note that $m'(\epsilon) \equiv (\mathrm{d}/\mathrm{d}\epsilon)m(\epsilon)$. The pressure p^* is given by

$$p^* = p_G - \frac{2\sigma}{a}, \qquad [7]$$

where p_G indicates the gas pressure, σ denotes the surface tension coefficient of the liquid/gas interface and a represents the average bubble radius. The quantities F_i ($i = 1, 2$) are specific body forces including mutual friction forces acting between the two phases. For more general dissipative forces see Geurst (1985) and Vreenegoor (1990). The quantities $\pi_i (i = 1, 2)$ which play a prominent role in [5] and [6], constitute the specific generalized momenta of, respectively, the liquid and the gas phase. They are determined by means of the following expressions for the generalized momentum densities $P_i (i = 1, 2)$ of the two phases:

$$P_1 = \rho_1\pi_1 = \rho_1 u_1 - \rho_L m(\epsilon)(u_2 - u_1) \qquad [8a]$$

and

$$P_2 = \rho_2\pi_2 = \rho_2 u_2 + \rho_L m(\epsilon)(u_2 - u_1). \qquad [8b]$$

While the first terms on the r.h.s. of [8a, b] evidently equal the "true" momentum densities of the liquid and the gas phase, the second terms having opposite signs represent the impulse or pseudo-momentum densities that are associated with the relative motion of the two phases. The fundamental quantity $m(\epsilon)$, which was called the *virtual-mass coefficient* in Geurst (1985), is introduced by means of the expression for the kinetic energy density K, viz.

$$K = \sum_{i=1}^{2} \tfrac{1}{2}\rho_i u_i^2 + \tfrac{1}{2}\rho_L m(\epsilon)(u_2 - u_1)^2. \qquad [9]$$

The last term in [9] represents the kinetic energy of the "backflow" velocity field induced in the liquid by the relative motion of the bubbles. Note that the part of K that is associated with the liquid is non-negative iff $m(\epsilon) \geqslant 0$. A thorough analysis (Geurst 1985, 1986; Wallis 1989) shows that the average total pressure p is given by

$$p = p^* + \tfrac{1}{2}\rho_L(1-\epsilon)^2\frac{\mathrm{d}}{\mathrm{d}\epsilon}E(\epsilon)(u_2 - u_1)^2, \qquad [10]$$

where $E(\epsilon)$ is the *exertia* introduced by Wallis (1989) according to

$$E(\epsilon) = \frac{m(\epsilon)}{1-\epsilon}. \qquad [11]$$

When the body forces $F_i (i = 1, 2)$ vanish, a uniform two-phase flow is *marginally stable* with respect to small perturbations iff (Geurst 1985; Prosperetti & Satrape 1990)

$$m(\epsilon) = \tfrac{1}{2}\epsilon(1-\epsilon)[\hat{m} - (\hat{m} + 2)\epsilon]. \qquad [12]$$

In the case of spherical bubbles ($\hat{m} = 1$) expression [12] reduces to [1].

3. DYNAMICS OF BUBBLE DISPERSION

Let us start with confining the attention to the special case of a *homogeneous* bubble dispersion, because it constitutes the simplest situation in which the dynamical effects associated with the impulse and the virtual mass of the bubbles are perceptible. Since the condition of homogeneity requires that the spatial derivatives of all quantities other than the pressure vanish, the kinematic equations [3] and [4] reduce to

$$\frac{\partial \epsilon}{\partial t} = \frac{\partial \rho_G}{\partial t} = \frac{\partial n}{\partial t} = 0. \qquad [13]$$

Various frames of reference may be selected for the dynamical analysis of a bubble dispersion. We consider here only (i) the laboratory frame of reference used in section 2 and (ii) the reference frame moving with the volume velocity of the bubbly mixture which is being employed by van Wijngaarden and his group.

(i) It will be clear from [8b] and [9] that in the laboratory frame of reference

$$P_2 = \left(\frac{\partial K}{\partial u_2}\right)_{u_1}. \qquad [14]$$

This is in accordance with Goldstein (1980, p. 563) and various other textbooks like Landau & Lifshitz (1962).

(ii) The volume velocity u_0 is defined by

$$u_0 = (1 - \epsilon)u_1 + \epsilon u_2. \qquad [15]$$

It is easily verified that the generalized momentum densities $P_i (i = 1, 2)$ given by [8a, b] may be expressed in terms of u_0 and u_2 in the following way:

$$P_1 = \rho_1 u_0 - \rho_L[E(\epsilon) + \epsilon](u_2 - u_0) \qquad [16a]$$

and

$$P_2 = \rho_2 u_2 + \rho_L E(\epsilon)(u_2 - u_0). \qquad [16b]$$

In the case of an homogeneous bubble dispersion the dynamic equations [5] and [6], accordingly, take the form

$$\frac{\partial}{\partial t}\{\rho_1 u_0 - \rho_L[E(\epsilon) + \epsilon](u_2 - u_0)\} = -(1 - \epsilon)\frac{\partial p}{\partial x} + \rho_1 F_1 \qquad [17]$$

and

$$\frac{\partial}{\partial t}\{\rho_2 u_2 + \rho_L E(\epsilon)(u_2 - u_0)\} = -\epsilon\frac{\partial p}{\partial x} + \rho_2 F_2. \qquad [18]$$

With a view to the use of a frame of reference moving with the volume velocity the kinetic energy density K given by [9] may be written as [see Kok (1988)]:

$$K = \tfrac{1}{2}\rho u_0^2 + (\rho_G - \rho_L)\epsilon u_0(u_2 - u_0) + \tfrac{1}{2}[\rho_2 + \rho_L k(\epsilon)](u_2 - u_0)^2, \qquad [19]$$

where $\rho = \rho_1 + \rho_2$ and

$$k(\epsilon) = \frac{m(\epsilon)}{(1-\epsilon)^2} + \frac{\epsilon^2}{1-\epsilon} = \frac{E(\epsilon) + \epsilon^2}{1-\epsilon}. \qquad [20]$$

It is easily recognized that the quadratic form for the kinetic energy density associated with the liquid which appears in [19] is non-negative definite iff

$$(1 - \epsilon)k(\epsilon) - \epsilon^2 \geqslant 0. \qquad [21]$$

That condition is clearly equivalent to $m(\epsilon) \geqslant 0$. The quantity $M(\epsilon) = k(\epsilon)/\epsilon$ is considered by van Wijngaarden (1991) as the proper virtual-mass coefficient of a bubble dispersion.

We remark that

$$\left(\frac{\partial K}{\partial u_2}\right)_{u_0} = -\rho_L \epsilon u_0 + \rho_2 u_2 + \rho_L k(\epsilon)(u_2 - u_0). \qquad [22]$$

Comparing [16b] and [22] we conclude that

$$P_2 \neq \left(\frac{\partial K}{\partial u_2}\right)_{u_0}. \qquad [23]$$

Indeed, the equality sign holds in [23] iff $u_0 = 0$ and $k(\epsilon) = -\epsilon$. That last expression, however, is not allowed according to [21]. A Galilean tranformation applied to [22] shows that the r.h.s. contains contributions from the true momenta of both the liquid and the gas phase, while P_2 comprises the true momentum of the bubbles only.

It follows from [20] that, in general, $k(\epsilon)$ does not vanish when the backflow energy may be disregarded, i.e. when $m(\epsilon) \equiv 0$ like in the case of stratified flow. This is caused by the fact that $k(\epsilon)$ contains an additional contribution originating from the kinetic energy of the mass-averaged motion of the liquid. The quantity $k(\epsilon)/\epsilon$ is accordingly not an acceptable candidate for a virtual-mass coefficient. van Wijngaarden (1991), however, puts it in the forefront as the proper quantity for the analysis of virtual-mass effects.

Eliminating the pressure gradient between [17] and [18] and taking $F_i = -g + F_i'$ $(i = 1, 2)$, where g denotes the gravitational acceleration, we obtain

$$\frac{\partial}{\partial t}[\rho_2 u_2 + \rho_L k(\epsilon)(u_2 - u_0) - \rho_L \epsilon u_0] = \epsilon(\rho_L - \rho_G)g + \epsilon(\rho_G F_2' - \rho_L F_1'). \qquad [24]$$

Note that the expression between square brackets equals $(\partial K/\partial u_2)_{u_0}$ by virtue of [22]. The corresponding evolution equation for u_0 is also obtained from [17] and [18]. It reads

$$\frac{\partial}{\partial t}\{[\rho + \chi(\epsilon)\rho_2]u_0\} = -\frac{\partial p}{\partial x} - \rho g + \rho_1 F_1' + \rho_2 F_2' + \chi(\epsilon)\left(-\epsilon\frac{\partial p}{\partial x} - \rho_2 g + \rho_2 F_2'\right), \qquad [25]$$

where

$$\chi(\epsilon) = \frac{1 - \left(\frac{\rho_G}{\rho_L}\right)}{\left[\frac{E(\epsilon)}{\epsilon}\right] + \left(\frac{\rho_G}{\rho_L}\right)}. \qquad [26]$$

More generally, going beyond the limitation set by the condition of homogeneity, we obtain by eliminating the pressure gradient between the original equations of motion [5] and [6] and then taking ρ_G = constant the following equation generalizing [24]:

$$\frac{\partial}{\partial t}[\epsilon\rho_G u_2 + \rho_L k(\epsilon)(u_2 - u_0)] + \frac{\partial}{\partial x}\{u_2[\epsilon\rho_G u_2 + \rho_L k(\epsilon)(u_2 - u_0)]\}$$

$$+\frac{\partial}{\partial x}\left\{-\tfrac{1}{2}\rho_L \epsilon^2 \frac{\mathrm{d}}{\mathrm{d}\epsilon}\left[\frac{k(\epsilon)}{\epsilon}\right](u_2 - u_0)^2\right\} - \epsilon\frac{\partial}{\partial t}(\rho_L u_0) = \epsilon(\rho_L - \rho_G)g + \epsilon(\rho_G F_2' - \rho_L F_1'). \qquad [27]$$

This equation is, as far as inertial effects are concerned, very similar to equation [41] of Biesheuvel & Gorissen (1990). The third term on the l.h.s., however, seems to be new. It is remarkable that the expression in braces in the third term on the l.h.s. of [27], playing the role of an "effective" pressure, vanishes when $k(\epsilon)$ satisfies [2] in the case of marginal stability. The fact, however, that according to the analysis of Biesheuvel & Gorissen (1990) the "effective" pressure vanishes when the spatial fluctuations of the bubble velocities are disregarded, does *not* imply that, conversely, the bubble velocity fluctuations have to disappear when the "effective" pressure vanishes.

It is interesting to note in this connection that van Wijngaarden & Biesheuvel (1988) erroneously omit the spatial derivative of the "effective" pressure in their [3. 4]. Because their stability analysis is based on that equation, the resulting stability criterion, which is reproduced in van Wijngaarden (1991) by inequality [7], is not correct. Note that the quantity $M(\epsilon)$ appearing in that criterion is, according to our notation, equal to $k(\epsilon)/\epsilon$. In fact, omission of the "effective" pressure term implies, by virtue of [27], that $M(\epsilon)$ is taken constant. van Wijngaarden & Biesheuvel (1988) therefore assume implicitly the validity of the marginal stability criterion [2]. Inequality [7] of van Wijngaarden (1991), however, imposes an additional restriction on the functional form of $M(\epsilon)$ which clearly is at variance with [2], when the mutual friction between the two phases is taken into account.

We emphasize that bubble interactions and bubble velocity fluctuations should be distinguished clearly. In fact, bubble interactions remain effective also in the hypothetical case, where the spatial fluctuations of the bubble velocities vanish and the local velocities of the bubbles are accordingly all equal, because the velocity fields induced by the separate bubbles in the surrounding liquid affect each other also in that case. It should be realised in that context that a random distribution of bubbles with *equal* bubble velocities is the situation envisaged by Kok (1988) and Biesheuvel & Spoelstra (1989) for analysing the effect of bubble interactions on the virtual mass of a bubble dispersion.

It may be added that the frame of reference moving with the volume velocity u_0 is in general *not* an inertial frame of reference. That is the reason for the appearance of the last term on the l.h.s. of [27]: it represents an apparent force accounting for a possible non-uniform motion of the reference frame.

Like van Wijngaarden (1991) is doing in his note, Biesheuvel & Gorissen (1990) are considering $k(\epsilon)/\epsilon$ as the virtual-mass coefficient of a bubble dispersion [more specifically $m_0(\epsilon)$ which equals $2k(\epsilon)/\epsilon$]. That, however, is not allowed for physical reasons, as we pointed out earlier. In fact, [27] is not the equation of motion of the gas phase. It results as we have seen, from combining the equations of motion of *both* phases, the gas phase *and* the liquid phase.

van Wijngaarden (1991) introduces the quantity I which he calls the "Kelvin impulse" of a bubble dispersion. He takes $\rho_G = 0$ and considers a moment at which $u_0 = 0$. Assuming that the bubble velocities are equal, he derives that

$$I = \left(\frac{\partial K}{\partial u_2}\right)_{u_0} = \rho_L k(\epsilon)(u_2 - u_0). \qquad [28]$$

Since in that particular case the impulse density of the bubble dispersion is, according to [16b], given by

$$P_2 = \rho_L E(\epsilon)(u_2 - u_0), \qquad [29]$$

we conclude that, in accordance with [23],

$$I \neq P_2. \qquad [30]$$

van Wijngaarden (1991), however, identifies his quantity I with the impulse density of a bubble dispersion. That is clearly not correct.

4. VOID-FRACTION DISTURBANCE AS A KINEMATIC WAVE

The simple form [2] taken by the marginal stability criterion deserves an explanation. Let us consider a homogeneous bubble dispersion in which the void fraction ϵ suffers a one-dimensional disturbance. It will be natural to assume that the mass density of the gas and the average bubble radius remain unaffected by the disturbance. The three kinematic equations [3] and [4] then require in the case of a uniformly propagating void-fraction disturbance that

$$u_2 = \text{const}, \quad u_1 = u_2 - \frac{c}{1-\epsilon}, \qquad [31]$$

where c denotes a constant (Geurst & Vreenegoor 1988). By virtue of conditions [31], each kinematic equation reduces to the equation

$$\left(\frac{\partial}{\partial t} + u_2 \frac{\partial}{\partial x}\right)\epsilon = 0, \qquad [32]$$

which has the general solution $\epsilon = \epsilon(x - u_2 t)$. It follows from [31] also that $u_1 = u_1(x - u_2 t)$. It is immediately inferred from [31] that

$$u_0 = u_2 - c. \qquad [33]$$

The volume velocity u_0 is, accordingly, a quantity independent of position and time. Under the very special conditions considered a reference frame moving with the volume velocity therefore appears like an inertial frame of reference.

It is shown in detail by Geurst & Vreenegoor (1988) that, body forces being disregarded and [31] being fulfilled, the two dynamic equations [5] and [6] both reduce to [32] iff the virtual-mass coefficient $m(\epsilon)$ satisfies [1]. The marginal stability criterion expressed by [1] or [2] accordingly constitutes a necessary and sufficient condition for the *identical* fulfilment of the dynamic equations in the case, where the kinematic equations are already satisfied according to [31] and [32]. It may therefore be considered as a critical condition for the kinematic wave character of void-fraction disturbances.

It should be emphasized that the analysis in this section is confined to the non-dissipative behaviour of a void-fraction disturbance in a bubbly flow without buoyancy. When gravity and mutual friction forces are taken into account, as in [5] and [6], the condition for *stability* of a uniform bubbly flow might be modified. In particular, the propagation velocity of a void-fraction disturbance might deviate from the drift velocity of the bubbles (cf. Prosperetti & Satrape 1990).

Because the two dynamic equations both reduce to [32] under kinematic wave conditions, all equations derived from them have that property. Indeed, inspection shows that [27] reduces to [32] when the body forces on the r.h.s. are neglected, conditions [31] are fulfilled and $k(\epsilon)$ satisfies the marginal stability criterion [2]. It is easily recognized that, in the case of spherical bubbles, [2] is not only sufficient but also necessary for a reduction of [27] to the kinematic wave equation [32]. That result shows that the simple form [2] taken by the marginal stability criterion is closely related to the kinematic wave character of void-fraction disturbances.

It will be clear that the marginal stability criterion has nothing to do with the neglect of bubble interactions. In fact, it means that dynamic effects that might cause instabilities are suppressed. That is the reason, why the "effective" pressure appearing in [27] vanishes on account of [2]. The vanishing of the "effective" pressure might be achieved microscopically by a proper distribution of the positions and velocities of the bubbles. It would be interesting to determine such a local distribution of the bubbles.

5. CONCLUSION

It has been demonstrated that the expressions for the impulse and virtual mass of a bubble dispersion used by van Wijngaarden and his group are not acceptable from a physical point of view. The argument advanced by van Wijngaarden (1991) that the form $k(\epsilon) = \epsilon/2$ of the marginal stability criterion might be related to the neglect of bubble interactions is based on those expressions and is therefore not correct.

It turns out that an alternative stability criterion put forward by van Wijngaarden & Biesheuvel (1988) is incorrect because it has been obtained by erroneously omitting the "effective" pressure derived in section 3. The expression for the "effective" pressure seems to be new.

The analysis of void-fraction waves given in Geurst & Vreenegoor (1988) proved to be helpful in clarifying the physical background of the marginal stability criterion of the author. It appeared that the frame of reference employed by van Wijngaarden and his group, although unsuitable for the analysis of *dynamic* properties like the impulse and virtual mass of a bubble dispersion, might be used profitably for describing *kinematic* phenomena like void-fraction waves.

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